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Structure of fluctuation terms in the trace dynamics Ward identity

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Abstract
We give a detailed analysis of the anti-self-adjoint operator contribution to the fluctuation terms in the trace dynamics Ward identity. This clarifies the origin of the apparent inconsistency between two forms of this identity discussed in chapter 6 of our recent book on emergent quantum theory.

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1. Introduction

In our recent book *Quantum Theory as an Emergent Phenomenon* [1], we developed a classical dynamics of non-commuting matrix (or operator) variables, with cyclic permutation inside a trace used as the basic calculational tool. We argued that quantum theory is the statistical thermodynamics of this underlying theory, with canonical commutation/anticommutation relations, and unitary quantum dynamics, both consequences of a generalized equipartition theorem. We also argued that fluctuation or Brownian motion corrections to this thermodynamics lead to state vector reduction and the probabilistic interpretation of quantum theory. In our analysis of fluctuation corrections, we noted that an anti-self-adjoint driving term, coming from a self-adjoint contribution to the conserved charge $\tilde{C}$ for global unitary invariance, is needed to give a stochastic Schrödinger equation that actually reduces the state vector. However, we also encountered an apparent inconsistency when such an anti-self-adjoint driving term was present, in that this term did not flip sign appropriately in going from the equation for a fermion operator ψ to that for its adjoint $\psi^{\dagger}$. (See the discussion following equation (6.7*a*) in chapter 6 of [1].)

Our aim in this paper is to give a detailed analysis of the origin of this apparent inconsistency. We shall show that when details that were glossed over in the treatment of chapter 6 are taken into account, the different forms of the Ward identity are always consistent, but in certain cases the anti-self-adjoint driving terms tend to cancel. Specifically, we shall show the following. (1) A self-adjoint term in $\tilde{C}$ appears when a fixed operator is used in the construction of the fermion kinetic terms, but cancels when this operator is elevated to a dynamical variable. (2) In the generic case when a self-adjoint term is present in $\tilde{C}$, the conjugate canonical momentum p_ψ is no longer equal to $\psi^\dagger$. The two equations that are analogues of the equations for ψ and $\psi^\dagger$ in equation (6.7a) of [1] are then equations for ψ and p_ψ, and the fact that the anti-self-adjoint driving term has the same sign in both equations is no longer an inconsistency. (3) In special cases where there are degrees of freedom with conventional fermion kinetic structure, that couple only indirectly through bosonic variables to fermion degrees of freedom that give rise to the self-adjoint term in $\tilde{C}$, the problem noted in chapter 6 of [1] reappears. However, it is not an inconsistency in the Ward identities, but rather an indication that the τ terms, which were neglected in the approximations leading to emergent quantum theory, must play a role. In other words, in this case, the anti-self-adjoint driving term in the stochastic equation cancels to the level of the terms neglected in our approximation scheme.

This paper is organized as follows. In section 2, we analyse two models for bilinear fermionic Lagrangians, focusing on the structure associated with the appearance of a self-adjoint component in $\tilde{C}$. In section 3, we derive the corresponding Ward identities analogous to equation (6.7a) of [1]. In section 4, we discuss the implications of these results for the apparent inconsistency discussed in chapter 6 of [1], leading to the conclusions briefly stated above.

2. Analysis of models for bilinear fermionic Lagrangians

In this section we analyse two models for bilinear fermionic Lagrangians. The first, which generalizes the model developed in equations (2.17)–(2.21) of [1], involves a fixed matrix A_{rs} in the fermion kinetic term and develops a self-adjoint contribution to $\tilde{C}$. In the second, the matrix A_{rs} is elevated to a bosonic dynamical variable, in which case its contribution to $\tilde{C}$ exactly cancels the self-adjoint fermionic contribution to $\tilde{C}$.

The first model that we consider is based on the bilinear fermionic trace Lagrangian

$$\mathbf{L} = \mathrm{Tr} \sum_{ra,sa,sb \in F} q_{ra}^\dagger A_{rs} (\dot{q}_{sa} + q_{sb} B_{ab}) + \text{bosonic}, \tag{1a}$$

where the notation $\in F$ (which will be suppressed henceforth) indicates a sum over fermionic degrees of freedom q_{ra}, labelled by the composite index ra, and where the purely bosonic terms are not explicitly shown. Here A_{rs} is a fixed bosonic matrix and B_{ab} is a bosonic operator (a generalized gauge potential). Recalling our adjoint convention that for fermionic χ_1, χ_2, we have $(\chi_1\chi_2)^\dagger = -\chi_2^\dagger \chi_1^\dagger$, we see that the Lagrangian of equation (1a) is real up to a total time derivative which vanishes in the expression for the trace action, provided that

$$A_{rs}^\dagger = A_{sr}, \qquad B_{ab}^\dagger = -B_{ba}. \tag{1b}$$

Introducing the canonical momentum defined by

$$p_{sa} = \frac{\delta \mathbf{L}}{\delta \dot{q}_{sa}} = \sum_r q_{ra}^\dagger A_{rs}, \tag{2a}$$

the trace Hamiltonian defined by

$$\mathbf{H} = \mathrm{Tr} \sum_{sa} p_{sa} \dot{q}_{sa} - \mathbf{L} \tag{2b}$$

has fermionic terms given explicitly by

$$\mathbf{H} = -\mathrm{Tr} \sum_{sab} p_{sa} q_{sb} B_{ab}. \tag{2c}$$

From this we find the equations of motion

$$\dot{q}_{sa} = -\frac{\delta \mathbf{H}}{\delta p_{sa}} = -\sum_b q_{sb} B_{ab},$$
$$\dot{p}_{sa} = -\frac{\delta \mathbf{H}}{\delta q_{sa}} = \sum_b B_{ba} p_{sb}, \qquad (2d)$$

where in the first line we have used the cyclic permutation rule for fermionic variables, $\mathrm{Tr}\,\chi_1\chi_2 = -\mathrm{Tr}\,\chi_2\chi_1$.

Although the trace Lagrangian in equation (1*a*) involves the fixed non-commutative matrix A_{rs}, this does not appear explicitly in the trace Hamiltonian, and so the conditions for global unitary invariance of the theory are fulfilled. Consequently, there is a conserved Noether charge $\tilde{C}$ given by

$$\tilde{C} = \tilde{C}_{\mathrm{F}} + \tilde{C}_{\mathrm{B}}. \qquad (3a)$$

The bosonic part $\tilde{C}_{\mathrm{B}}$ is given by

$$\tilde{C}_{\mathrm{B}} = \sum_{r \in B} [q_r, p_r] \qquad (3b)$$

and is anti-self-adjoint in the generic case with the bosonic canonical variables q_r, p_r either both self-adjoint or both anti-self-adjoint. The fermionic part $\tilde{C}_{\mathrm{F}}$ is given by

$$\tilde{C}_{\mathrm{F}} = -\sum_{ra} (q_{ra} p_{ra} + p_{ra} q_{ra}), \qquad (4a)$$

and by using equation (2*a*) we find that $\tilde{C}_{\mathrm{F}}$ has a self-adjoint part $\tilde{C}_{\mathrm{F}}^{\mathrm{sa}}$ given explicitly by

$$\tilde{C}_{\mathrm{F}}^{\mathrm{sa}} = \frac{1}{2}\left(\tilde{C}_{\mathrm{F}} + \tilde{C}_{\mathrm{F}}^{\dagger}\right) = \frac{1}{2}\sum_{rsa} \left[A_{rs}, q_{sa} q_{ra}^{\dagger}\right]. \qquad (4b)$$

Using the equations of motion of equation (2*d*), we find that $\tilde{C}_{\mathrm{F}}$ has the time derivative

$$\dot{\tilde{C}}_{\mathrm{F}} = -\sum_{rab} [B_{ab}, p_{ra} q_{rb}] = -\sum_{rsab} \left[B_{ab}, q_{sa}^{\dagger} A_{sr} q_{rb}\right], \qquad (4c)$$

from which we see that $\dot{\tilde{C}}_{\mathrm{F}}$ is anti-self-adjoint, as required by the fact that it must cancel against the anti-self-adjoint contribution coming from $\dot{\tilde{C}}_{\mathrm{B}}$. Thus, the self-adjoint part of $\tilde{C}_{\mathrm{F}}$ given in equation (4*b*) is separately conserved. This can also be verified directly by using equation (2*d*) and its adjoint, together with equation (1*b*), as follows:

$$\dot{\tilde{C}}_{\mathrm{F}}^{\mathrm{sa}} = \frac{1}{2}\sum_{rsa} \left[A_{rs}, \dot{q}_{sa} q_{ra}^{\dagger} + q_{sa} \dot{q}_{ra}^{\dagger}\right]$$
$$= -\frac{1}{2}\sum_{rsab} \left[A_{rs}, q_{sb} B_{ab} q_{ra}^{\dagger} + q_{sa} B_{ab}^{\dagger} q_{rb}^{\dagger}\right]$$
$$= -\frac{1}{2}\sum_{rsab} \left[A_{rs}, q_{sb} B_{ab} q_{ra}^{\dagger} - q_{sa} B_{ba} q_{rb}^{\dagger}\right] = 0. \qquad (4d)$$

In writing the Ward identities to be discussed in the next section, several auxiliary quantities related to the above discussion will be needed. First, we will need a self-adjoint operator Hamiltonian H, the trace of which gives the trace Hamiltonian $\mathbf{H} = \mathrm{Tr}\,H$. This can be constructed from the self-adjoint part of any cyclic permutation of the factors in equation (2*c*), and so is not unique. We will adopt the simplest choice, with fermionic terms given by the expression

$$H = H^\dagger = -\frac{1}{2}\sum_{sab}(p_{sa}q_{sb}B_{ab} + B_{ab}p_{sa}q_{sb}). \tag{5a}$$

Because this is a function only of the dynamical variables but not of the fixed bosonic matrix A_{rs}, under a unitary transformation of the dynamical variables $p_{sa} \to U^\dagger p_{sa}U, q_{sb} \to U^\dagger q_{sb}U$, $B_{ab} \to U^\dagger B_{ab}U$, the Hamiltonian H of equation (5*a*) has the attractive feature of being unitary covariant, $H \to U^\dagger HU$. An alternative expression for the operator Hamiltonian H, that yields the same trace Hamiltonian $\mathbf{H}$, is given by

$$\frac{1}{2}\sum_{sab}[q_{sb}B_{ab}p_{sa} + (q_{sb}B_{ab}p_{sa})^\dagger] = \frac{1}{2}\sum_{sab}\left[q_{sb}B_{ab}p_{sa} + \sum_{ru}A_{sr}q_{ra}B_{ba}p_{ub}A^{-1}_{us}\right], \tag{5b}$$

but since this explicitly involves both A_{sr} and its inverse A^{-1}_{us}, it is a less natural choice than equation (5*a*) (it is not a unitary covariant, as well as being less tractable), and we will not use it in the discussion that follows.

We will also need to evaluate the anticommutator expression

$$\mathrm{i}_{\mathrm{eff}}\tilde{C}_{\mathrm{eff}} \equiv \tfrac{1}{2}\{\tilde{C}, \mathrm{i}_{\mathrm{eff}}\} \equiv -\hbar(1 + \mathcal{K} + \mathcal{N}), \tag{5c}$$

where $\mathrm{i}_{\mathrm{eff}}$ and $\hbar$ are the effective imaginary unit and Planck constant given by the ensemble expectation $\langle\tilde{C}\rangle_{\mathrm{AV}} = \mathrm{i}_{\mathrm{eff}}\hbar$ (see equation (4.11*b*) of [1]), and where $-\hbar\mathcal{K}$ and $-\hbar\mathcal{N}$ are, respectively, the *c*-number and operator parts of the fluctuating part of $\mathrm{i}_{\mathrm{eff}}\tilde{C}_{\mathrm{eff}}$. At this point, we introduce the specialization that the fixed matrix A_{rs} commutes with $\mathrm{i}_{\mathrm{eff}}$,

$$[\mathrm{i}_{\mathrm{eff}}, A_{rs}] = 0, \tag{6a}$$

as a consequence of which, by the cyclic identities, we have

$$\mathrm{Tr}\,\mathrm{i}_{\mathrm{eff}}\tilde{C}^{\mathrm{sa}}_{\mathrm{F}} = \frac{1}{2}\sum_{rsa}\mathrm{Tr}[\mathrm{i}_{\mathrm{eff}}, A_{rs}]q_{sa}q^\dagger_{ra} = 0. \tag{6b}$$

This implies that it is consistent to ignore the self-adjoint part of $\tilde{C}$ in forming the canonical ensemble. Since ensemble expectations are then functions only of $\mathrm{i}_{\mathrm{eff}}$, a second consequence of equation (6*a*) is that

$$\langle\tilde{C}^{\mathrm{sa}}_{\mathrm{F}}\rangle_{\mathrm{AV}} = \frac{1}{2}\sum_{rsa}\left[A_{rs}, \langle q_{sa}q^\dagger_{ra}\rangle_{\mathrm{AV}}\right] = 0, \tag{6c}$$

which implies that even in the presence of $\tilde{C}^{\mathrm{sa}}_{\mathrm{F}}$, we can still define an effective imaginary unit by the phase of the ensemble expectation of $\tilde{C}$.

Returning to equation (5*c*), we now specify conditions to make the separation into terms $\mathcal{K}$ and $\mathcal{N}$ unique. In [1], a normal ordering prescription in the emergent field theory was invoked, but here we stay within the underlying trace dynamics and impose the natural conditions that $\mathcal{K}$ and $\mathcal{N}$ are, respectively, the *c*-number part, and the traceless part, of equation (5*c*). Then as a consequence of equation (6*b*), the self-adjoint part $\tilde{C}^{\mathrm{sa}}_{\mathrm{F}}$ makes a vanishing contribution to $\mathcal{K}$, which therefore is a real number, while the operator $\mathcal{N}$ receives an anti-self-adjoint contribution $\mathcal{N}^{\mathrm{asa}}$ given by

$$-\hbar\mathcal{N}^{\mathrm{asa}} = \mathrm{i}_{\mathrm{eff}}\tilde{C}^{\mathrm{sa}}_{\mathrm{eff}}. \tag{6d}$$

Let us turn now to a second model for the bilinear fermionic trace Lagrangian, which has a similar structure to that of equation (1*a*), but with the matrix A_{rs} now itself a dynamical variable. Since $\dot{A}_{rs}$ is no longer zero, to get a trace Lagrangian that is real up to time derivative terms, we must redefine the fermion kinetic part of equation (1a) according to

$$\mathbf{L} = \mathrm{Tr} \sum_{rsab} \left[q_{ra}^{\dagger} A_{rs} (\dot{q}_{sa} + q_{sb} B_{ab}) + \frac{1}{2} q_{ra}^{\dagger} \dot{A}_{rs} q_{sa} \right] + \text{bosonic}. \tag{7a}$$

The canonical momentum p_{ra} is unchanged in form, but now there is a bosonic canonical momentum P_{rs} conjugate to A_{rs} given by

$$P_{rs} = \frac{\delta \mathbf{L}}{\delta \dot{A}_{rs}} = -\frac{1}{2} \sum_{a} q_{sa} q_{ra}^{\dagger} + \frac{\delta \mathbf{L}_{\mathrm{bosonic}}}{\delta \dot{A}_{rs}}. \tag{7b}$$

Since $\left(q_{sa} q_{ra}^{\dagger}\right)^{\dagger} = -q_{ra} q_{sa}^{\dagger}$, the canonical momentum P_{rs} now has the adjoint behaviour $P_{rs}^{\dagger} \neq P_{sr}$, and as a consequence, the contribution of the canonical pair A_{rs}, P_{rs} to $\tilde{C}$ is no longer anti-self-adjoint, but instead has a self-adjoint part

$$\left(\sum_{rs} [A_{rs}, P_{rs}] \right)^{\mathrm{sa}} = \frac{1}{2} \sum_{rs} \left[A_{rs}, P_{rs} - P_{sr}^{\dagger} \right] = -\frac{1}{2} \sum_{rsa} \left[A_{rs}, q_{sa} q_{ra}^{\dagger} \right], \tag{7c}$$

which exactly cancels the self-adjoint fermionic contribution of equation (4b). Thus, when the matrix A_{rs} is elevated to a dynamical variable, the Noether charge $\tilde{C}$ is purely anti-self-adjoint.

3. Fluctuation terms in the trace dynamics Ward identities

We proceed now to work out the implications of the Lagrangian of equation (1a) for the trace dynamics Ward identities. To make contact with equation (6.7a) of [1], we shall not need the most general form of these identities, but only the statement that the quantities $\mathcal{D} q_{ra\,\mathrm{eff}}$ and $\mathcal{D} p_{ra\,\mathrm{eff}}$ vanish when sandwiched between general polynomial functions of the 'eff' projections of the dynamical variables, and averaged over the zero source canonical ensemble. The ensemble equilibrium distribution is given by $\rho = Z^{-1} \exp(-\lambda\, \mathrm{Tr}\, \mathrm{i}_{\mathrm{eff}} \tilde{C} - \tau \mathbf{H})$, with λ and τ parameters characterizing the ensemble, and with Z (the 'partition function') the ensemble normalizing factor. For a fermionic x_u, the expression $\mathcal{D} x_{u\,\mathrm{eff}}$ is given by

$$\mathcal{D} x_{u\,\mathrm{eff}} = -\tau \dot{x}_{u\mathrm{eff}} \mathrm{Tr}\, \tilde{C} \mathrm{i}_{\mathrm{eff}} W_{\mathrm{eff}} + [\mathrm{i}_{\mathrm{eff}} W_{\mathrm{eff}}, x_{u\mathrm{eff}}] + \sum_{s,\ell} \omega_{us} \epsilon_{\ell} \left(W_s^{R\ell} \frac{1}{2} \{\tilde{C}, \mathrm{i}_{\mathrm{eff}}\} W_s^{L\ell} \right)_{\mathrm{eff}}. \tag{8}$$

Here ω_{us} is a matrix with element -1 when s is the label of the variable x_s conjugate to x_u, and 0 otherwise, and W is a general self-adjoint bosonic polynomial in the dynamical variables. The quantities in the final term are defined by writing the variation of W when the variable x_s is varied (which we denote by $\delta_{x_s} W$) in the form

$$\delta_{x_s} W = \sum_{\ell} W_s^{L\ell} \delta x_s W_s^{R\ell}, \tag{9a}$$

where ℓ is a composite index that labels each monomial in the polynomial W, as well as each occurrence of x_s in the respective monomial term. In this notation we have

$$\frac{\delta \mathbf{W}}{\delta x_s} = \sum_{\ell} \epsilon_{\ell} W_s^{R\ell} W_s^{L\ell}, \tag{9b}$$

with ϵ_{ℓ} the grading factor appropriate to $W_s^{R\ell}$ and to $W_s^{L\ell} x_s$ (which must both be of the same grade since we have defined W to be bosonic).

We will apply the above expressions when W is taken as the Hamiltonian H with fermionic terms given by equation (5a). For the fermionic variations of H we find

$$\delta H = -\frac{1}{2}\sum_{sab}(\delta p_{sa} q_{sb} B_{ab} + p_{sa}\delta q_{sb} B_{ab} + B_{ab}\delta p_{sa} q_{sb} + B_{ab} p_{sa}\delta q_{sb}), \qquad (10a)$$

from which we can read off the factors $W_s^{L\ell}$, $W_s^{R\ell}$ and ϵ_ℓ needed in equation (8). For example, when x_u is the variable q_{sa}, the index s in equation (8) labels the canonical conjugate variable p_{sa}. Referring to equation (9a), we see that the composite index ℓ takes the respective values 1 and 2, b for the two factor orderings in equation (10a), with

$$\begin{aligned} W_s^{R1} &= -\frac{1}{2}\sum_b q_{sb} B_{ab}, \qquad & W_s^{L1} &= 1, \qquad & \epsilon_1 &= -1, \\ W_s^{R2,b} &= -\frac{1}{2} q_{sb}, & W_s^{L2,b} &= B_{ab}, & \epsilon_{2,b} &= -1. \end{aligned} \qquad (10b)$$

The corresponding expressions when x_u is the variable p_{sa} have a similar structure that can easily be read off from the terms in equation (10b) in which δq_{sb} appears. Assembling the various pieces of equation (8), and using equations (5c) and (9b), we get the following two formulae:

$$\begin{aligned} \mathcal{D} q_{ra\,\mathrm{eff}} &= -\tau \dot{q}_{ra\mathrm{eff}}\,\mathrm{Tr}(\tilde{C}^{\mathrm{asa}} + \tilde{C}^{\mathrm{sa}}) \mathrm{i}_{\mathrm{eff}} H_{\mathrm{eff}} + \mathrm{i}_{\mathrm{eff}}[H_{\mathrm{eff}}, q_{ra\,\mathrm{eff}}] \\ &\quad - \hbar(1+\mathcal{K})\dot{q}_{ra\,\mathrm{eff}} + \frac{1}{2}\hbar\sum_b \left(q_{rb}\{B_{ab}, \mathcal{N}^{\mathrm{sa}} + \mathcal{N}^{\mathrm{asa}}\}\right)_{\mathrm{eff}}, \\ \mathcal{D} p_{ra\,\mathrm{eff}} &= -\tau \dot{p}_{ra\,\mathrm{eff}}\,\mathrm{Tr}(\tilde{C}^{\mathrm{asa}} + \tilde{C}^{\mathrm{sa}}) \mathrm{i}_{\mathrm{eff}} H_{\mathrm{eff}} + \mathrm{i}_{\mathrm{eff}}[H_{\mathrm{eff}}, p_{ra\,\mathrm{eff}}] \\ &\quad - \hbar(1+\mathcal{K})\dot{p}_{ra\,\mathrm{eff}} - \frac{1}{2}\hbar\sum_b (\{B_{ba}, \mathcal{N}^{\mathrm{sa}} + \mathcal{N}^{\mathrm{asa}}\} p_{rb})_{\mathrm{eff}}, \end{aligned} \qquad (11a)$$

where we have explicitly separated $\tilde{C}$ and $\mathcal{N}$ into self-adjoint (superscript sa) and anti-self-adjoint (superscript asa) parts. Taking the adjoint of the first of these equations, and remembering that $B_{ab}^{\dagger} = -B_{ba}$, we also get for comparison the formula

$$\begin{aligned} (\mathcal{D} q_{ra\,\mathrm{eff}})^{\dagger} &= -\tau \dot{q}^{\dagger}_{ra\,\mathrm{eff}}\,\mathrm{Tr}(\tilde{C}^{\mathrm{asa}} - \tilde{C}^{\mathrm{sa}}) \mathrm{i}_{\mathrm{eff}} H_{\mathrm{eff}} + \mathrm{i}_{\mathrm{eff}}\left[H_{\mathrm{eff}}, q^{\dagger}_{ra\,\mathrm{eff}}\right] \\ &\quad - \hbar(1+\mathcal{K})\dot{q}^{\dagger}_{ra\,\mathrm{eff}} - \frac{1}{2}\hbar\sum_b \left(\{B_{ba}, \mathcal{N}^{\mathrm{sa}} - \mathcal{N}^{\mathrm{asa}}\} q^{\dagger}_{rb}\right)_{\mathrm{eff}}. \end{aligned} \qquad (11b)$$

4. Discussion

The formulae of equations (11a) and (11b), which so far involve no approximations, are the analogues within the model of equation (1a) of the similar formulae given in equation (6.7a) of [1]. They differ from equation (6.7a) in a number of respects.

(1) First, the structure of the term involving $\mathcal{N}$ is different from what appears in [1] because the simplest choice for the self-adjoint operator Hamiltonian H, when the matrices A_{rs} and B_{ab} are non-trivial operators, has the structure of equation (5a), in both terms of which p_{sa} stands to the left of q_{sb}. When $B_{ab} = \mathrm{i}m\delta_{ab}$, corresponding to a mass term, this reduces to $H = -\mathrm{i}m\sum_{sa} p_{sa} q_{sa}$, which when $A_{rs} = \delta_{rs}$ further reduces to $H = -\mathrm{i}m\sum_{sa} q^{\dagger}_{sa} q_{sa}$, which does not have the commutator structure assumed on an ad hoc basis in equation (6.6) of [1]. As a result, in equation (11b) the creation operator $q^{\dagger}_{rb}$ automatically stands on the right, and the assumption made in [1] that $\mathcal{N}$ is normal ordered is not necessary.

(2) In the treatment here, we have based the separation of the fluctuation term into $\mathcal{K}$ and $\mathcal{N}$ terms on a decomposition into c-number and traceless parts, rather than an invocation of normal ordering. As a result, we saw that $\mathcal{K}$ receives no anti-self-adjoint contribution, and so is a real (rather than a complex) number. In terms of the discussion of chapter 6 of [1], this means that the model of equation (1a) does not lead to energy-driven reduction, which requires a nonzero imaginary part of $\mathcal{K}$. Localization-driven reduction, which arises from the anti-self-adjoint part of $\mathcal{N}$, is still allowed.

(3) In the generic case when A_{rs} is not equal to δ_{rs} in any sector, the canonical momentum p_{sa} is not the same as the adjoint $q_{sa}^{\dagger}$. So even when the τ terms in equations (11a) and (11b) are dropped, there is no contradiction arising from the fact that $\mathcal{N}^{\mathrm{asa}}$ appears in the second equation of equation (11a) and in equation (11b) with opposite signs. Thus, in the generic case, the inconsistency discussed following equation (6.7a) of [1] is not present.

(4) However, there is a specialization of equations (11a) and (11b) in which an analogue of the problem noted in [1] persists. Suppose that we divide the fermionic degrees of freedom q_{ra} into two classes I and II, based on the value of the index r, and take A_{rs} to be block diagonal within the two classes. For class I degrees of freedom, we take A_{rs} to be non-trivial, so that $p_{ra} \neq q_{ra}^{\dagger}$. For class II degrees of freedom, we take $A_{rs} = \delta_{rs}$, so that $p_{ra} = q_{ra}^{\dagger}$. Then if we restrict equations (11a) and (11b) to r values for class II degrees of freedom, we see that the second equation in equation (11a) has the same structure as equation (11b), except that the terms involving $\tilde{C}^{\mathrm{sa}}$ and $\mathcal{N}^{\mathrm{asa}}$ both have opposite signs in the two equations. Hence, taking the difference between the second equation in equation (11a), and equation (11b), we get for r in class II,

$$\mathcal{D}q_{ra\,\mathrm{eff}}^{\dagger} - (\mathcal{D}q_{ra\,\mathrm{eff}})^{\dagger} = -2\tau \dot{q}_{ra\,\mathrm{eff}}^{\dagger}\,\mathrm{Tr}\,\tilde{C}^{\mathrm{sa}} \mathrm{i}_{\mathrm{eff}} H_{\mathrm{eff}} - \hbar \sum_{b} \left(\{B_{ba}, \mathcal{N}^{\mathrm{asa}}\} q_{rb}^{\dagger}\right)_{\mathrm{eff}}. \tag{12}$$

This expression must vanish when inserted (sandwiched between polynomials in the variables) in canonical ensemble averages. Hence, in this case, which is a somewhat more general version of the model formulated in equation (6.6) of [1], the terms involving $\mathcal{N}^{\mathrm{asa}}$ must effectively average to be of the same order of magnitude as the τ terms, which were neglected in the approximation scheme of chapter 5 of [1]. There is no inconsistency in the Ward identities of equations (11a) and (11b), or in equation (12) that is derived from them, but in this case one cannot consistently drop the τ terms and reinterpret these equations as operator equations at the level of the emergent quantum theory.

To conclude, we have re-examined the apparent inconsistency arising from equation (6.7a) of [1], taking into account details that were not sufficiently carefully dealt with there. We see that in the generic case one can still use operator analogues of equations (11a) and (11b) as the basis for a state vector reduction model. But we have also seen that there is a tendency for the needed anti-self-adjoint driving term to cancel, suggesting some caution, and also more speculatively, suggesting a reason why one might expect the state vector reduction terms to be small corrections to the basic emergent Schrödinger equation.

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References

[1] Adler S L 2004 *Quantum Theory as an Emergent Phenomenon* (Cambridge: Cambridge University Press)